Confinement and bound states in QCD

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I revue the so called Wilson loop approach to bound state problem in QCD. I shall show how using appropriate path integral representations for the quark propagator in an external field it is possible to obtain corresponding path integral representations for various types of gauge invariant Green functions which have the important feature of involving the gauge field only trough Wilson loop correlators or their generalizations. Two different kinds of representations are used, one given in the form of a semi-relativistic expansion, the second completely relativistic of the Feynmann-Schwinger type. In this way starting from reasonable ansatz on the non perturbative part of the Wilson correlator one can obtain: expressions for the semi relativistic (spin dependent and momentum dependent) $q\bar{q}$ and $3q$ potentials, a "second order" $q\bar{q}$ Bethe-Salpeter equation and and a related Dyson-Schwinger equation. I shall concentrate on the three quark potential for which new controversial results have been obtained by lattice numerical simulations and on a three dimensional reduction of the BS equation obtained in the form of the eigenvalue equation of of a squared or a usual mass operator. We shall report on a numerical resolution of such equations which seems to give a comprehensive reproduction of the entire meson spectrum with the exception of light pseudo-scalar bound states for which a complete four dimensional treatment should be necessary.

1 Green Functions

The single quark, the quark-antiquark and the three quark gauge invariant Green functions can be written as

$$G^{\rm gi}(x-y) = \langle 0|{\rm T}U(y,x)\psi(x)\overline{\psi}(y)\overline{\psi}(y)|0\rangle = {\rm Tr}_{\rm C}\langle U(y,x)S(x,y;A)\rangle\,, \qquad (1)$$

$$\begin{aligned} G^{\rm gi}(x_1,x_2;y_1,y_2) &= \frac{1}{3}\langle 0|{\rm T}\psi_2^c(x_2)U(x_2,x_1)\psi_1(x_1)\overline{\psi}_1(y_1)U(y_1,y_2)\overline{\psi}_2^c(y_2)|0\rangle = \\ &= \frac{1}{3}{\rm Tr}_{\rm C}\langle U(x_2,x_1)S_1(x_1,y_1;A)U(y_1,y_2)\tilde{S}_2(y_2,x_2;-\tilde{A})\rangle \qquad (2)\end{aligned}$$

(the two quarks are supposed to have a different flavor, otherwise an annihilation term should be added to the last term)

$$\begin{aligned} &G^{\rm gi}(x_1,x_2,x_3;y_1,y_2,y_3) = \frac{1}{3!}\varepsilon_{a_1a_2a_3}\varepsilon_{b_1b_2b_3}\langle 0|{\rm T}\,U^{a_3c_3}(x_M,x_3)U^{a_2c_2}(x_M,x_2) \\ &U^{a_1c_1}(x_M,x_1)\psi_{3c_3}(x_3)\psi_{2c_2}(x_2)\psi_{1c_1}(x_1)\overline{\psi}_{1d_1}(y_1)\overline{\psi}_{2d_2}(y_2)\overline{\psi}_{3d_3}(y_3) \\ &U^{d_1b_1}(y_1,y_M)U^{d_2b_2}(y_2,y_M)U^{d_3b_3}(y_3,y_M)|0\rangle = \frac{1}{3!}\varepsilon_{a_1a_2a_3}\varepsilon_{b_1b_2b_3} \\ &\Big\langle \big(U(x_M,x_1)S_1(x_1,y_1;A)U(y_1,y_M)\big)^{a_1b_1}\big(U(x_M,x_2)S_2(x_2,y_2;A)\end{aligned}$$

$$U(y_2,y_M)\big)^{a_2b_2}\big(U(x_M,x_3)S_3(x_3,y_3;A)U(y_3,y_M)\big)^{a_3b_3}\rangle\,. \tag{3}$$

To avoid complicate indices in the above equations we have identified the various type of functions simply by their arguments; furthermore ψ^c denotes the charge-conjugate fields, and U the path-ordered gauge string (Schwinger string)

$$U(b,a) = \mathrm{P}\exp\left\{ig\int_a^b dx^\mu\, A_\mu(x)\right\}, \tag{4}$$

(the integration in (4) is along an arbitrary line joining a to b), the tilde and $\mathrm{Tr_C}$ denote the transposition and the trace over the color indices alone. The second equalities are the result of an explicit integration over the fermionic fields and the angle brackets denote average on the gauge variable alone (weighted in principle with the determinant $M_f(A)$ resulting from the integration). The quantities $S,\,\ldots S_3$ are the quark propagators in the external gauge field A^μ, which are defined by equations of the type (we shall suppress indices specifying the quarks, as a rule, when dealing with single quark quantities)

$$(i\gamma^\mu D_\mu - m)S(x,y;A) = \delta^4(x-y)\,. \tag{5}$$

Here m is supposed to hold an infinitesimal negative imaginary part ($m = m - i0$).

2 Quark-antiquark potential

By performing a Foldy–Wouthuysen transformation on Eq. (5) we can replace the the 4×4 Dirac type matrices $S(x,y;A)$ with a 2×2 Pauli propagator $K(\mathbf{x},\mathbf{y};t_\mathrm{f},t_\mathrm{i})$ satisfying a Schrödinger-like equation. Solving such equation by standard path-integral technique, we obtain

$$\begin{aligned}
&K(\mathbf{x},\mathbf{y};t_\mathrm{f},t_\mathrm{i}) = \int_{\mathbf{z}_1(t_\mathrm{i})=\mathbf{y}_1}^{\mathbf{z}_1(t_\mathrm{f})=\mathbf{x}_1} \mathcal{D}\mathbf{z}\mathcal{D}\mathbf{p} \\
&\exp\{i\int_{t_\mathrm{i}}^{t_\mathrm{f}} dt\,[\mathbf{p}\cdot\dot{\mathbf{z}} - m - \frac{\mathbf{p}^2}{2m} + \frac{\mathbf{p}^4}{8m^3}]\}\mathrm{T_s}\,\mathrm{P}\exp\{ig\int_\Gamma dz^\mu\, A_\mu(z) + \\
&+\frac{ig}{m_j}\int_\Gamma dz^\mu(S_j^l\hat{F}_{l\mu}(z) - \frac{1}{2m_j}S_j^l\varepsilon^{lkr}p^kF_{\mu r}(z) - \frac{1}{8m}D^\nu F_{\nu\mu}(z))\}\,.
\end{aligned} \tag{6}$$

Here Γ is the quark world line connecting $(\mathbf{y},t_\mathrm{i})$ with $(\mathbf{x},t_\mathrm{f})$ over which the path integral acts, $\mathrm{T_s}$ and P are time ordering prescriptions for the spin and the color matrices respectively. Furthermore, as usual $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu + ig[A^\mu,A^\nu]$, $\hat{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ and $D^\nu F_{\nu\mu} = \partial^\nu F_{\nu\mu} + ig[A^\nu,F_{\nu\mu}]$, $\varepsilon^{\mu\nu\rho\sigma}$ being the four-dimensional Ricci symbol.

Replacing (6) in (2) we obtain a path integral representation for a Pauli type $q\bar{q}$ propagator which can be compared with the corresponding expression in potential theory (see [1] for details). Then one find that, in order the interaction between the two particles can be described in terms of a potential, a function $V^{q\overline{q}}(\mathbf{z}_1, \mathbf{z}_2, \mathbf{p}_1, \mathbf{p}_2, \mathbf{S}_1, \mathbf{S}_2)$ must exist such that op to the order $\frac{1}{m^2}$

$$
\begin{aligned}
& i \ln W_{q\overline{q}} + i \sum_{j=1}^{2} \frac{ig}{m_j} \int_{\Gamma_j} dx^\mu \Bigg(S_j^l \, \langle\langle \hat{F}_{l\mu}(x) \rangle\rangle - \frac{1}{2m_j} S_j^l \varepsilon^{lkr} p_j^k \, \langle\langle F_{\mu r}(x) \rangle\rangle - \\
& \quad - \frac{1}{8m_j} \, \langle\langle D^\nu F_{\nu\mu}(x) \rangle\rangle \Bigg) - \frac{1}{2} \sum_{j,j'} \frac{ig^2}{m_j m_{j'}} \mathrm{T_s} \int_{\Gamma_j} dx^\mu \int_{\Gamma_{j'}} dx'^\sigma \, S_j^l \, S_{j'}^k \\
& \Bigg(\langle\langle \hat{F}_{l\mu}(x) \hat{F}_{k\sigma}(x') \rangle\rangle - \langle\langle \hat{F}_{l\mu}(x) \rangle\rangle \, \langle\langle \hat{F}_{k\sigma}(x') \rangle\rangle \Bigg) + \ldots = \int_{t_\mathrm{i}}^{t_\mathrm{f}} dt \, V^{q\overline{q}} \, , \quad (7)
\end{aligned}
$$

with the notation $\langle\langle f[A] \rangle\rangle = \langle W_{q\overline{q}} f[A] \rangle \, / \, \langle W_{q\overline{q}} \rangle$ and $W_{q\overline{q}}$ is the Wilson loop correlator defined by

$$
W_{q\overline{q}} = \frac{1}{3} \left\langle \mathrm{Tr_C P} \exp \left(ig \oint_{\Gamma_{q\overline{q}}} dx^\mu \; A_\mu(x) \right) \right\rangle . \quad (8)
$$

In Eq. (8) the integration loop $\Gamma_{q\overline{q}}$ is supposed to be made by the quark world line Γ_1, the antiquark world line Γ_2 described in reverse direction and the two Schwinger strings that close the curve and can be here taken equal time straight lines; the symbol P denotes color ordering along $\Gamma_{q\overline{q}}$.

Notice that the quantities $\langle\langle F_{\mu\nu}(z_j) \rangle\rangle$ and $\langle\langle F_{\mu\nu}(z_j) F_{\rho\sigma}(z'_j) \rangle\rangle$ can be expressed as functional derivatives of $i \ln W_{q\overline{q}}$ and so the potential is determined in principle by such quantity alone.

Eq. (7) can be reelaborated in various way. By expanding $i \ln W_{q\overline{q}}$ and the spin dependent terms in $\dot{z}_j$ it is possible to recast the resulting local coefficients in terms of static Wilson loop correlators (straight quark world-lines parallel to the time axis) with insertion of field strengths. Such expressions are particularly suitable for numerical simulation and to this aim have been successfully used [2].

To obtain analytical expressions today we have to relay on models for the evaluation of the Wilson correlator. The simplest model is the so called minimal area law model (MAL model) and consists in writing $i \ln W_{q\overline{q}}$ as the sum of its perturbative, an area and possibly a perimeter term

$$
i \ln W_{q\overline{q}} = (i \ln W_{q\overline{q}})_{\mathrm{pert}} + \sigma S_{\mathrm{min}} + \frac{1}{2} C P \quad (9)
$$

where S_{min} denotes the minimal surface enclosed by the loop $\Gamma_{q\overline{q}}$ and P length of the loop. In Eq. (9) the first quantity is supposed to give correctly the short range limit of the interaction and it is suggested by asymptotic freedom; the other two are supposed to represent the long range behavior and are suggested by pure lattice gauge theory and numerical simulations [3].

At the lowest order the perturbative term can be written as

$$i(\ln W_{q\overline{q}})_{\rm pert} = -\frac{2}{3}g^2 \oint dz^\mu \oint dz^{\nu\prime} D_{\mu\nu}(z-z') \tag{10}$$

$D_{\mu\nu}(z-z')$ being the free gauge propagator. For what concerns the area term it can be checked, by solving the appropriate Euler equations, that $S_{\rm min}$ can be replaced by the surface spanned by the straight lines joining equal time points on Γ_1 and Γ_2 up to $O(v^2)$. That is, we can write

$$S_{\rm min} \approx \int_{t_{\rm i}}^{t_{\rm f}} dt\ \sigma r \int_0^1 d\lambda\ [1-(\lambda\dot{\mathbf{z}}_{1\rm T} + (1-\lambda)\dot{\mathbf{z}}_{2\rm T})^2]^{\frac{1}{2}} \tag{11}$$

where $\dot{\mathbf{z}}_{j\rm T}$ denotes the *transversal part* of $\dot{\mathbf{z}}_j$, $\dot{z}^h_{j\rm T} = (\delta^{hk} - \hat{r}^h\hat{r}^k)\dot{z}^h_j$.

Replacing (10) and (11) in (9) and (7), by expanding in the velocities and other manipulations we obtain a semi-relativistic potential of the form $V^{q\overline{q}} = V^{q\overline{q}}_{\rm static} + V^{q\overline{q}}_{\rm sd} + V^{q\overline{q}}_{\rm vd}$, where $V^{q\overline{q}}_{\rm static} = -\frac{4}{3}\alpha_{\rm s}\frac{1}{r} + \sigma r + C$, $V^{q\overline{q}}_{\rm sd}$ and $V^{q\overline{q}}_{\rm vd}$ are certain complicate spin dependent and velocity dependent expressions which are reported in [1] but we do not reproduce here. Similar results can be obtained starting from more elaborate models [4].

3 Three quark potential

A semi-relativistic three quark potential can be obtained proceeding in a similar way on the three quark Green function (3). Instead of (7) this time we have

$$\begin{aligned} & i\ln W_{3q} + i\sum_{j=1}^3 \frac{ig}{m_j}\int_{\Gamma_j} dx^\mu \biggl(S^l_j \, \langle\langle \hat{F}_{l\mu}(x)\rangle\rangle - \frac{1}{2m_j} S^l_j \varepsilon^{lkr} p^k_j \, \langle\langle F_{\mu r}(x)\rangle\rangle - \\ & -\frac{1}{8m_j}\, \langle\langle D^\nu F_{\nu\mu}(x)\rangle\rangle \biggr) - \frac{1}{2}\sum_{j,j'} \frac{ig^2}{m_j m_{j'}} \mathrm{T_s} \int_{\Gamma_j} dx^\mu \int_{\Gamma_{j'}} dx'^\sigma\, S^l_j\, S^k_{j'} \\ & \biggl(\langle\langle \hat{F}_{l\mu}(x)\hat{F}_{k\sigma}(x')\rangle\rangle - \langle\langle \hat{F}_{l\mu}(x)\rangle\rangle\, \langle\langle \hat{F}_{k\sigma}(x')\rangle\rangle \biggr) = \int_{t_{\rm i}}^{t_{\rm f}} dt\, V^{3q} \end{aligned} \tag{12}$$

where now $\langle\langle f[A]\rangle\rangle = \langle W_{3q} f[A]\rangle / \langle W_{3q}\rangle$ and W_{3q} is the Wilson loop correlator for three quarks

$$\begin{aligned} W_{3q} = \frac{1}{3!}\varepsilon_{a_1a_2a_3}\varepsilon_{b_1b_2b_3} & \left[\mathrm{P}\exp\left(ig\int_{\overline{\Gamma}_1} dx^{\mu_1} A_{\mu_1}(x)\right)\right]^{a_1b_1} \\ \left[\mathrm{P}\exp\left(ig\int_{\overline{\Gamma}_2} dx^{\mu_2} A_{\mu_2}(x)\right)\right]^{a_2b_2} & \left[\mathrm{P}\exp\left(ig\int_{\overline{\Gamma}_3} dx^{\mu_3} A_{\mu_3}(x)\right)\right]^{a_3b_3} . \end{aligned} \tag{13}$$

In (13) a_j, b_j are color indices, $j = 1, 2, 3$ and $\overline{\Gamma}_j$ denote the curve made by the world lines Γ_j for the quark j joining y_j to x_j, a straight line on the surface $t = t_{\rm i}$ merging from an arbitrary fixed point y_M and arriving to y_j, another straight line on the surface $t = t_{\rm f}$ connecting the world line to a second fixed x_M.

A straightforward generalization of the arguments used in the quark-antiquark case suggests to write in place of (9) and (10)

$$i \ln W_{3q} = \frac{2}{3} g^2 \sum_{i<j} \int_{\Gamma_i} dx_i^\mu \int_{\Gamma_j} dx_j^\nu \; i D_{\mu\nu}(x_i - x_j) + \sigma S_{\rm min} + \frac{1}{3} C P \,, \tag{14}$$

where the perturbative term is taken at the lowest order in α_s and now $S_{\rm min}$ denotes the minimum among all the surfaces made by three sheets having the curves $\overline{\Gamma}_1$, $\overline{\Gamma}_2$ and $\overline{\Gamma}_3$ as contours and joining on a line Γ_M connecting $y_{\rm M}$ with $x_{\rm M}$ (the minimum is understood at fixed $\bar{\Gamma}_j$ as the surfaces and $\Gamma_{\rm M}$ change). Obviously, P denotes the total length of $\overline{\Gamma}_1$, $\overline{\Gamma}_2$ and $\overline{\Gamma}_3$.

As in the $q\overline{q}$ case, up to the second order in the velocities $S_{\rm min}$ can be replaced by the surface spanned by the straight lines joining the point $\mathbf{z}_M(t)$ at the time t with the equal time positions of the three quarks $\mathbf{z}_1(t)$, $\mathbf{z}_2(t)$ and $\mathbf{z}_3(t)$, $\mathbf{z}_M(t)$ being constructed according to the following rule: if no angle in the triangle made by $\mathbf{z}_1(t)$, $\mathbf{z}_2(t)$ and $\mathbf{z}_3(t)$ exceeds 120^0 (configuration I), $\mathbf{z}_M(t)$ coincides with the point inside the triangle which sees the three sides under the same angle 120^0; if one of the three angles in the triangle is $\geq 120^0$ (configuration II), $\mathbf{z}_M(t)$ coincides with the corresponding vertex, let us say $\mathbf{z}_{\bar{j}}(t)$.

The final result is of the form $V^{3q} = V^{3q}_{\rm stat} + V^{3q}_{\rm sd} + V^{3q}_{\rm vd}$, where $V^{3q}_{\rm stat} = \sum_{j<l} \left(-\frac{2}{3} \frac{\alpha_s}{r_{jl}} \right) + \sigma(r_1 + r_2 + r_3) + C$ (with $\mathbf{r}_j = \mathbf{z}_j - \mathbf{z}_M$, $\mathbf{r}_{jl} = \mathbf{r}_j - \mathbf{r}_l \equiv \mathbf{z}_j - \mathbf{z}_l$), while $V^{3q}_{\rm sd}$ and $V^{3q}_{\rm vd}$ are spin dependent and momentum depend expressions of order $1/m^2$ for which again I refer to [1].

We observe also that the short range part in V^{3q}, coming from the first term in (14) is of a pure two body type: in fact it is identical to the electromagnetic potential among three equally charged particles but for the color group factor 2/3.

The three sheet definition of $S_{\rm min}$ given above is suggested by a direct generalization of Wilson's original argument for $W_{q\bar{q}}$, based on the lattice formulation. It is adopted by the majority of the authors and it is said to correspond to an Y flux tube configuration. However an alternative possibility has also been considered [5]. This consists in setting

$$S_{\rm min} = \frac{1}{2} \sigma (S_{12} + S_{23} + S_{31}) \,, \tag{15}$$

where now S_{ij} denotes the minimal surface delimited $\overline{\Gamma}_i$ and $\overline{\Gamma}_j$ and the factor $\frac{1}{2}$ is introduced in order to reproduce the $q\bar{q}$ case when two quark world-lines collapse. Assumption (15) is said to correspond to a Δ flux tube configuration. In this case even the long range part of the three quark potential is the sum of purely two body potential and one can write simply $V_{3q} = \frac{1}{2}(V_{12}^{q\bar{q}} + V_{23}^{q\bar{q}} + V_{31}^{q\bar{q}})$, $V_{ij}^{q\bar{q}}$ being equal to the quark antiquark potential relative to the couple ij as defined by (7).

If we restrict ourself to the linear rising part (the confining part) of the static potentials alone we can write for the two configurations

$$V_Y^{\rm conf} = \sigma(r_1 + r_2 + r_3)\,, \tag{16}$$

$$V_\Delta^{\rm conf} = \frac{1}{2}\sigma(r_{12} + r_{23} + r_{31})\,. \tag{17}$$

Then, simple geometrical considerations show that

$$V_Y^{\rm conf} \leq V_\Delta^{\rm conf} \leq \frac{2}{\sqrt{3}} V_Y^{\rm conf}\,, \tag{18}$$

where the equal sign in the first step holds when the three quarks are aligned, in the second one when they form an equilateral triangle; the second one being the situation in which the difference between the two expressions is maximal (note, however that $\frac{2}{\sqrt{3}} \sim 1.155$). Until recently the various attempts to discriminate the two expressions by evaluating numerically $\ln W_{3q}$ for a static loop have been unsuccessful. Presently preliminary results obtained by Bali et al. [2] for the equilateral arrangement are claimed to support Δ configuration. On the contrary the very accurate estimates reported by Matsufuru at this conference [6] are definitely in favour of the Y configuration. Indeed it seems that, due to the large errors occurring in the results reported in Ref. [2] at the large distances, these can be reasonably reconciled with those of Ref. [6] simply by an appropriate choice of the constant C (private communication by Suganuma). If this were confirmed, the Y configuration should be considered established.

4 Bethe-Salpeter and Dyson-Schwinger equation

Full relativistic bound state equations can be obtained along similar lines using a covariant representation for the solution of (5).

To this aim it is convenient to rewrite the "first order" propagator $S(x,y;A)$ in terms of a "second order" one

$$S(x,y;A) = (i\gamma^\nu D_\nu + m)\Delta^\sigma(x,y;A)\,, \tag{19}$$

$\Delta^\sigma(x,y;A)$ being defined by

$$(D_\mu D^\mu + m^2 - \frac{1}{2} g\,\sigma^{\mu\nu} F_{\mu\nu})\Delta^\sigma(x,y;A) = -\delta^4(x-y)\,, \tag{20}$$

with $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$.

After replacing (19) in (2), using an appropriate derivative it is possible to take the differential operator out of the angle brackets and write [7]

$$G^{\rm gi}(x_1, x_2; y_1, y_2) = -(i\gamma_1^\mu \bar\partial_{1\mu} + m_1)(i\gamma_2^\nu \bar\partial_{2\nu} + m_2) H^{\rm gi}(x_1, x_2; y_1, y_2)\,; \quad (21)$$

a similar expression can be given for $G^{\rm gi}(x-y)$, having set

$$H^{\rm gi}(x_1, x_2; y_1, y_2) = -\frac{1}{3} {\rm Tr_C} \langle U(x_2, x_1) \Delta_1^\sigma(x_1, y_1; A) U(y_1, y_2) \tilde\Delta_2^\sigma(x_2, y_2; -\tilde A)\rangle\,,$$
$$H^{\rm gi}(x-y) = i{\rm Tr_C}\langle U(y,x)\Delta^\sigma(x,y;A)\rangle. \quad (22)$$

For the second order propagator we have the Feynmann-Schwinger representation

$$\Delta^\sigma(x,y;A) = -\frac{i}{2}\int_0^\infty ds \int_y^x \mathcal{D}z \exp[-i\int_0^s d\tau \frac{1}{2}(m^2 + \dot z^2)]$$
$$\mathcal{S}_0^s {\rm P}\exp[ig\int_0^s d\tau \dot z^\mu A_\mu(z)]\,, \quad (23)$$

where the world-line connecting y to x is specified in the four-dimensional language by $z^\mu = z^\mu(\tau)$, in terms of an additional parameter τ, and $\mathcal{S}_0^s = {\rm T}\exp\Big[-\frac{1}{4}\int_0^s d\tau \sigma^{\mu\nu}\frac{\delta}{\delta S^{\mu\nu}(z)}\Big]$ and $\delta S^{\mu\nu} = dz^\mu \delta z^\nu - dz^\nu \delta z^\mu$ (the functional derivative being defined through an arbitrary deformation, $z \to z + \delta z$, of the world-line).

Substituting (23) into (22), we obtain covariant path integral representations for $H^{\mathbf{gi}}(x_1, x_2; y_1, y_2)$ and $H^{\mathbf{gi}}(x-y)$. These representations involve again the gauge field only trough the Wilson correlator $W_{q\overline{q}}$ for the four point function and a similar quantity W_q for the two point function. W_q is obtained replacing in (8) the loop $\Gamma_{q\overline{q}}$ by a second loop made by the world-line connecting y to x closed by the appropriate Schwinger string [7].

For the Wilson loop correlator we can make the ansatz (9 -11)in the center of mass system even in the relativistic case, after taking $C = 0$ (since in a relativistic treatment the perimeter term can be completely reabsorbed in a mass renormalization). Notice that in such a case in general (11) is only an approximation, however, if rewritten in an appropriate way, it seems to be a significant one. In fact, beside to have the correct semi-relativistic limit,(11) becomes exact in important geometrical situations [7] like those of a flat Wilson loop or of uniform rotatory motion.

From the above path integral representations and by an appropriate recurrence method, an inhomogeneous Bethe-Salpeter equation and a Dyson-Schwinger equation can be derived for two other second order functions
$H(x_1, x_2, y_1, y_2)$ and $H(x-y)$, which are simply related to $H^{\rm gi}(x_1, x_2, y_1, y_2)$

and $H^{\rm gi}(x-y)$, reduce to them in the limit of vanishing $x_1-x_2,\ y_1-y_2$ or $x-y$ and are completely equivalent for what concerns the determination of bound states, effective masses, etc.

In the momentum space, the corresponding homogeneous BS-equation can be written (in a 4×4 matrix representation)

$$\Phi_P(k) = -i\int \frac{d^4u}{(2\pi)^4}\ \hat{I}_{ab}\big(k-u, \frac{1}{2}P+\frac{k+u}{2}, \frac{1}{2}P-\frac{k+u}{2}\big) \\ \hat{H}_1\big(\frac{1}{2}P+k\big)\sigma^a\Phi_P(u)\sigma^b\hat{H}_2\big(-\frac{1}{2}P+k\big)\,, \qquad (24)$$

where $\Phi_P(k)$ denotes an appropriate wave function, $\sigma^0=1$, $a=0,\mu\nu$ and the center of mass frame has to be understood (i.e. $P=(m_B,\mathbf{0})$, m_B being the bound state mass). Similarly, the DS-equation can be written also

$$\hat{\Gamma}(k) = \int\frac{d^4l}{(2\pi)^4}\ \hat{I}_{ab}\Big(k-l;\frac{k+l}{2},\frac{k+l}{2}\Big)\sigma^a\hat{H}(l)\,\sigma^b\ . \qquad (25)$$

$\hat{\Gamma}(k)$ being the irreducible self-energy, defined by $\hat{H}(k)=\hat{H}_0(k)+i\hat{H}_0(k)\hat{\Gamma}(k)\hat{H}(k)$.

Notice that in principle (24) and (25) are exact equations. However the kernels $\hat{I}_{ab}$ are generated in the form of an expansion in $\alpha_{\rm s}$ and the string tension σ. At the lowest order in both such constants, we have explicitly

$$\hat{I}_{0;0}(Q;p,p') = 16\pi\frac{4}{3}\alpha_{\rm s}p^\alpha p'^\beta\hat{D}_{\alpha\beta}(Q)+$$
$$+4\sigma\int d^3\zeta e^{-i\mathbf{Q}\cdot\zeta}|\zeta|\epsilon(p_0)\epsilon(p'_0)\int_0^1 d\lambda\{p_0^2p_0'^2-[\lambda p'_0\mathbf{p}_{\rm T}+(1-\lambda)p_0\mathbf{p}'_{\rm T}]^2\}^{\frac{1}{2}}$$
$$\hat{I}_{\mu\nu;0}(Q;p,p') = 4\pi i\frac{4}{3}\alpha_{\rm s}(\delta^\alpha_\mu Q_\nu-\delta^\alpha_\nu Q_\mu)p'_\beta\hat{D}_{\alpha\beta}(Q)-$$
$$-\sigma\int d^3\zeta\ e^{-i\mathbf{Q}\cdot\zeta}\epsilon(p_0)\frac{\zeta_\mu p_\nu-\zeta_\nu p_\mu}{|\zeta|\sqrt{p_0^2-\mathbf{p}_{\rm T}^2}}p'_0$$
$$\hat{I}_{0;\rho\sigma}(Q;p,p') = -4\pi i\frac{4}{3}\alpha_{\rm s}p^\alpha(\delta^\beta_\rho Q_\sigma-\delta^\beta_\sigma Q_\rho)\hat{D}_{\alpha\beta}(Q)+$$
$$+\sigma\int d^3\zeta\ e^{-i\mathbf{Q}\cdot\zeta}p_0\frac{\zeta_\rho p'_\sigma-\zeta_\sigma p'_\rho}{|\zeta|\sqrt{p_0'^2-\mathbf{p}_{\rm T}'^2}}\epsilon(p'_0)$$
$$\hat{I}_{\mu\nu;\rho\sigma}(Q;p,p') = \pi\frac{4}{3}\alpha_{\rm s}(\delta^\alpha_\mu Q_\nu-\delta^\alpha_\nu Q_\mu)(\delta^\alpha_\rho Q_\sigma-\delta^\alpha_\sigma Q_\rho)\hat{D}_{\alpha\beta}(Q) \qquad (26)$$

where in the second and in the third equation $\zeta_0=0$ has to be understood.

5 Reduction of the BS equation and spectrum.

To find the $q\overline{q}$ spectrum, in principle one should solve first (25) and use the resulting propagator in (24). In practice this turns out to be a difficult task and one has to resort to the three dimensional equation which can be obtained from (24) by the so called instantaneous approximation. This consists in replacing $\hat{H}_j(p)$ in (24) with the free quark propagator $\frac{-i}{p^2-m_j^2}$ and the kernel $\hat{I}_{ab}$ with $\hat{I}_{ab}^{\rm inst}(\mathbf{k},\mathbf{k}')$ obtained from $\hat{I}_{ab}$ setting $k_0 = k_0' = \frac{m_2}{m_1+m_2}\frac{w_1+w_1'}{2} - \frac{m_1}{m_1+m_2}\frac{w_2+w_2'}{2}$ with $w_j = \sqrt{m_j^2+\mathbf{k}^2}$ and $w_j' = \sqrt{m_j^2+\mathbf{k}'^2}$.

The reduced equation takes the form of the eigenvalue equation for a squared mass operator, $M^2 = M_0^2 + U$, with $M_0 = \sqrt{m_1^2+\mathbf{k}^2} + \sqrt{m_2^2+\mathbf{k}^2}$ and

$$\langle \mathbf{k}|U|\mathbf{k}'\rangle = \frac{1}{(2\pi)^3}\sqrt{\frac{w_1+w_2}{2w_1w_2}}\,\hat{I}_{ab}^{\rm inst}(\mathbf{k},\mathbf{k}')\sqrt{\frac{w_1'+w_2'}{2w_1'w_2'}}\sigma_1^a\sigma_2^b\,. \tag{27}$$

The quadratic form of the above equation obviously derives from the second order character of the formalism we have used.

Alternatively, in more usual terms, one can look for the eigenvalue of the mass operator or center of mass hamiltonian $H_{\rm CM} \equiv M = M_0 + V$ with V defined by $M_0V + VM_0 + V^2 = U$. Neglecting, consistently, the second order term V can be obtained from U simply by the kinematic replacement $\sqrt{\frac{(w_1+w_2)(w_1'+w_2')}{w_1w_2w_1'w_2'}} \to \frac{1}{2\sqrt{w_1w_2w_1'w_2'}}$.

In ref. [8] we have evaluated the spectrum using both the operator M^2 and $H_{\rm CM}$, including the hyperfine terms in the potentials but omitting the spin-orbit ones. The numerical procedure we have followed was very simple. It consisted in solving first the eigenvalue equation for the zero order hamiltonian $H_0^{\rm CM} = M_0 - \frac{4}{3}\frac{\alpha_{\rm s}}{r} + \sigma r$ by the Rayleigh-Ritz method, using the three-dimensional harmonic oscillator basis and diagonalising a 30×30 matrix. Then we have evaluated the quantities $\langle\psi_\nu|H_{\rm CM}|\psi_\nu\rangle$ and $\langle\psi_\nu|M^2|\psi_\nu\rangle$ for the eigenfunctions ψ_ν obtained in the first step. Since the quantity $\langle V^2\rangle$ was not completely negligible (ranging e. g. between few tens and 150 MeV for $c\bar{c}$), we had to use slight different values of the adjustable parameters in the linear and in the quadratic formulation. We have kept the light quark masses fixed in both cases on typical current values, $m_u = m_s = 10\,{\rm MeV}$, $m_s = 200\,{\rm MeV}$. Then for the calculations based on $H_{\rm CM}$ we have assumed $m_c = 1.40\,{\rm GeV}$, $m_b = 4.81\,{\rm GeV}$, $\alpha_{\rm s} = 0.363$ and $\sigma = 0.175\,{\rm GeV}^2$, taken essentially from the semi-relativistic fits. On the contrary for the calculations based on M^2 we have used a running coupling constant of the form

$$\alpha_{\rm s}(\mathbf{Q}) = \frac{\mathbf{4}\pi}{(\mathbf{11} - \frac{\mathbf{2}}{\mathbf{3}}\mathbf{N}_{\rm f})\ln\frac{\mathbf{Q^2}}{\mathbf{\Lambda^2}}} \tag{28}$$

cut at a maximum adjustable value $\alpha_{\rm s}(0)$ and with $N_{\rm f} = 4$, $\Lambda = 200\,{\rm MeV}$. Furthermore we have chosen $\alpha_{\rm s}(0) = 0.35$ and $\sigma = 0.2\,{\rm GeV}^2$ in order to reproduce the correct $J/\Psi - \eta_c$ separation and the Regge trajectory slope;

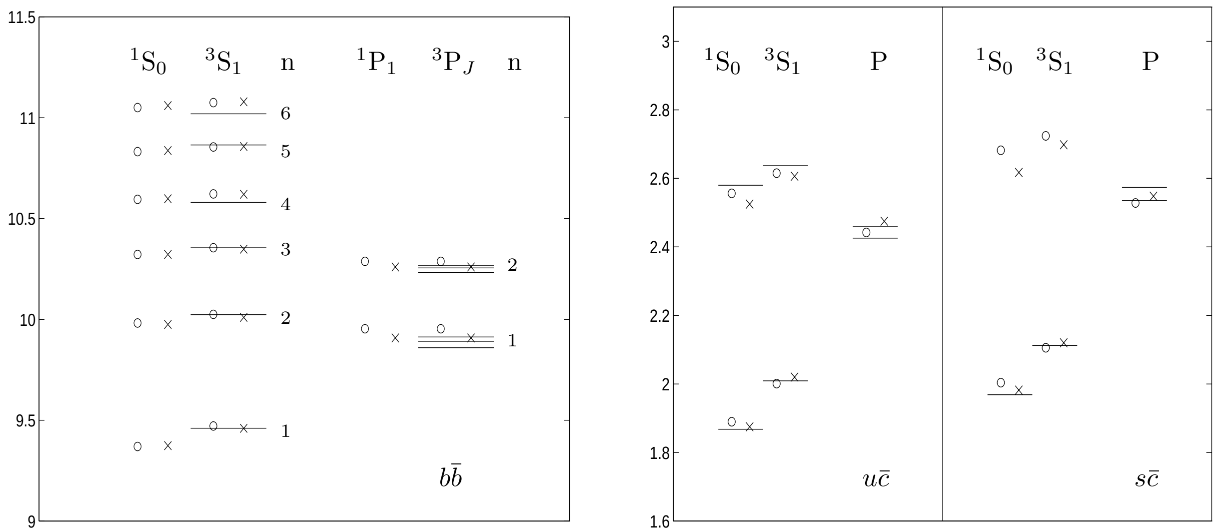

Figure 1: Heavy-heavy and light-heavy quarkonium spectra.

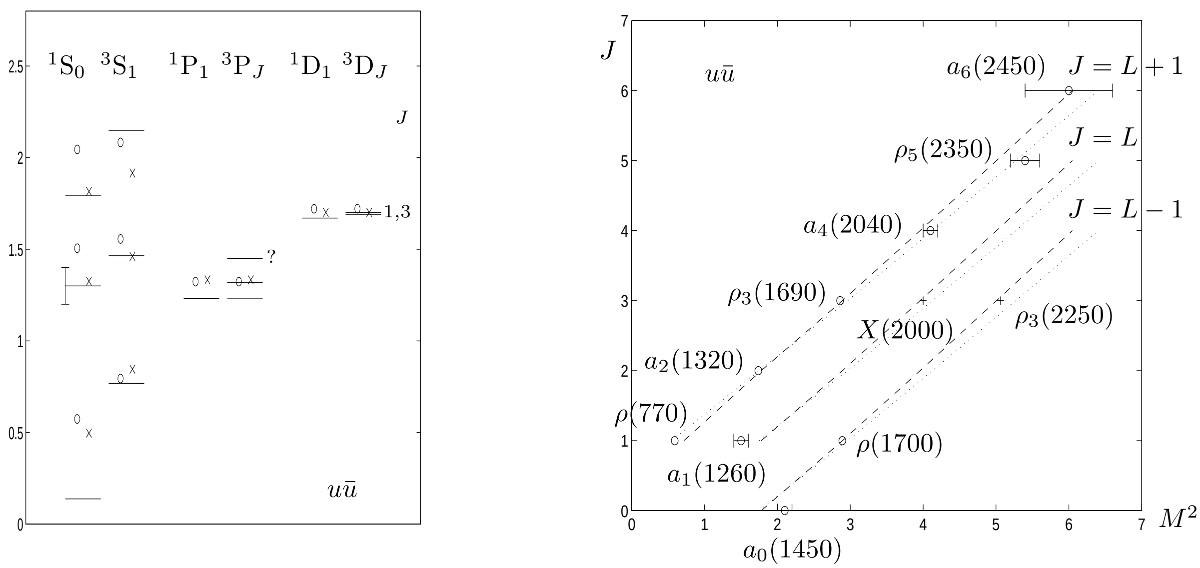

Figure 2: $u\bar{u}$ spectrum and Regge trajectories.

$m_c = 1.394\,\mathrm{GeV}$ and $m_b = 4.763\,\mathrm{GeV}$ in order to obtain exactly the masses of J/Ψ and Υ.

In both cases, on the whole, the agreement with the data is good, not only for bottonium and charmonium (as in ordinary potential models), but also for the light-light and light-heavy systems which are here essentially parameter free. Examples are reported in figs. 1 and 2. Circlet and dotted lines refer to the linear formulation, crosses and broken lines to the quadratic one. The only serious disagreement concerns the light pseudo-scalar mesons related to the chiral symmetry breaking problem. In fact in this case, as well known, a strict connection should exist between wave function in (24) and irreducible self energy in (25) and the use of the free propagator can not be correct.

6 References

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